NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS ADVANCED

2024 Year 12 Course Assessment Task 4 (Trial HSC Examination)
Monday, 19 August 2024

General instructions

- Working time – 3 hours.
  (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.

SECTION I

- Mark your answers on the answer grid provided (on page 29)

SECTION II

- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: ....

Class (please ✔)

◯ 12MAA.1 – Miss Chaussivert

◯ 12MAX.1 – Miss J. Kim

◯ 12MAX.2 – Mrs Bhamra

◯ 12MAX.3 – Miss Lee

Marker's use only.

QUESTION	1-10	11-14	15-17	18-20	21-23	24-25	26-29	**Total**
MARKS	/10	/14	/13	/13	/16	/18	/16	/100

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided (labelled as page 29).

Questions **Marks**

1. What is the solution to the equation $\log_3(2x-5) = 2$? **1**

(A) $x = 1$ (B) $x = \dfrac{13}{2}$ (C) $x = \dfrac{11}{2}$ (D) $x = 7$

2. Evaluate $\displaystyle\lim_{x\to 7} \frac{x^2+5x-84}{x-7}$ **1**

(A) -5 (B) 0 (C) 12 (D) 19

3. For what values of a does the equation $ax^2 + 5x + a = 0$ have no solution? **1**

(A) $a > 0$

(B) $a = \frac{5}{2}$

(C) $-\frac{5}{2} < a < \frac{5}{2}$

(D) $a < -\frac{5}{2}$ or $a > \frac{5}{2}$

4. A function f is given by the rule **1**

$$f(x) = \begin{cases} (x-2)^2 + 1, & x \le 2 \\ 4x - 7, & x > 2 \end{cases}$$

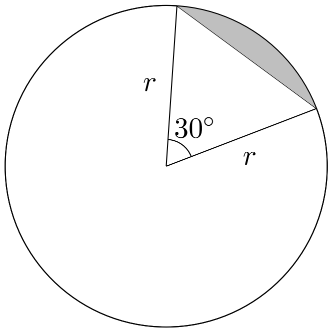
What is the value of $f\Big(f(0)\Big)$?

(A) 5 (B) -7 (C) -35 (D) 13

5. The first three terms of an arithmetic series are $5, 9,$ and 13. **1**
What is the 15th term of the series?

(A) 61 (B) 66 (C) 495 (D) 585

6. The diagram shows a circle with radius r centimetres and a sector which subtends an angle of $30°$ **1**



If the area of the shaded segment is 1.7 cm^2, which of the following is closest to the radius of the circle?

(A) 0.12 cm (B) 0.34 cm (C) 12.00 cm (D) 144.08 cm

7. Which of the following is an expression for $\displaystyle\int \left(\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right) dx$? **1**

(A) $2\sec x + c$

(B) $4\tan x \sec^2 x + c$

(C) $2\tan x + c$

(D) $2x + 2\sec x + c$

8. What is the domain and range of the function $y = \dfrac{1}{\sqrt{x-9}}$? **1**

(A) $D = \{x : x \geq 9\}$ and $R = \{y : y > 0\}$

(B) $D = \{x : x > 9\}$ and $R = \{y : y > 0\}$

(C) $D = \{x : -\infty \leq x \leq \infty\}$ and $R = \{y : -\infty \leq y \leq \infty\}$

(D) $D = \{x : x \leq -3 \text{ or } x \geq 3\}$ and $R = \{y : y < 0\}$

9. Olivia and Sarah are participating in a fitness program. The probability that Olivia will complete the program is 0.8, and the probability that Sarah will complete the program is 0.7. **1**

What is the probability that only one of them will complete the program?

(A) 0.14 (B) 0.24 (C) 0.38 (D) 0.56

10. Which of the following is the derivative of $e^{3^x x^6}$. **1**

(A) $3^x x^6 e^{3^x x^6}$

(B) $3^x x^6 e^{\left(3^x x^6 - 1\right)}$

(C) $3^x x^5 \left(6 + x \log_e 3\right) e^{3^x x^6}$

(D) $3^x x^5 \left(3^x x^6 + 6x^5 \left(\log_e 3\right) 3^x\right) e^{3^x x^6}$

Examination continues overleaf...

Section II

90 marks
Attempt Question 11 to 29
Allow approximately 2 hours and 45 minutes for this section

Write your answers in the space provided.

Question 11 (3 marks)

(a) Sketch the graph of $y = |x - 1|$ for $x \in [-4, 4]$. **1**

(b) Using the sketch from part (i) or otherwise, solve **2**

$$|x - 1| = 2x + 4$$

Question 12 (4 marks)

Given $f(x) = \sqrt{x^2 - 9}$ and $g(x) = x + 5$

(a) Find positive integers c and d such that $f\Big(g(x)\Big) = \sqrt{(x+c)(x+d)}$ **2**

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(b) State the domain for which $f\Big(g(x)\Big)$ is defined. **2**

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Question 13 (4 marks)

(a) Show that $\dfrac{d}{dx}\left(x \operatorname{cosec} x\right) = \operatorname{cosec} x - x \cot x \operatorname{cosec} x$ **3**

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(b) Hence find: **1**

$$\int 2\Big(\operatorname{cosec} x\,(1 - x\cot x)\Big)\,dx$$

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Examination continues overleaf. . .

Question 14 (3 marks)

Find the equation of the normal to the curve $y = \left(x + \frac{2}{x}\right)^2$ at the point where $x = 2$. **3**
Write the equation of the normal using general form.

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Question 15 (3 marks)

A and B are two events such that $P(A) = 0.3$, $P(B) = 0.2$ and $P(A|B) = 0.5$

(a) Show that $P(A \cup B) = 0.4$. **2**

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(b) Find the value of **1**

$$P\left(\overline{A} \cap \overline{B}\right)$$

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Question 16 (4 marks)

The discrete random variable X has the following distribution:

x	0	1	2	3	4
$P(X=x)$	0.1	$5a^2$	0.2	0.2	$2.5+11a$

(a) Show that $a = -\frac{1}{5}$. **3**

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(b) Find the mean of the distribution. **1**

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Examination continues overleaf...

Question 17 (6 marks)

There are 4 red and 3 black discs in a bag. Ms Kim and Ms Lee are playing a game in which they take turns drawing a disc from the bag and then replacing it.

To win the game, Ms Kim must draw a red disc and for Ms Lee to win, she must draw a black disc. They continue taking turns until there is a winner. Ms Kim draws a disc first.

(a) Find the probability that Ms Kim wins on her first draw. **1**

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(b) Find the probability that Ms Kim wins in three or less of her turns. **2**

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(c) Find the probability that Ms Kim wins the game. **3**

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Question 18 (2 marks)

(a) Prove the following identity: **2**

$$\frac{\cos\theta}{1+\sin\theta}+\frac{1+\sin\theta}{\cos\theta}=2\sec\theta.$$

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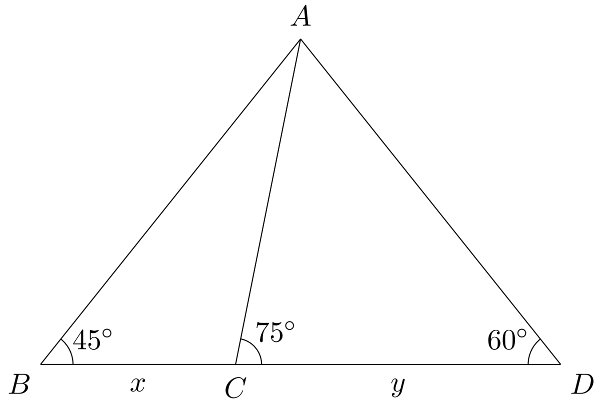
Examination continues overleaf...

Question 19 (7 marks)

Refer to the following diagram:



(a) Show that $\dfrac{x}{y} = \dfrac{\sqrt{3}}{2}$. **3**

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(b) Find the smallest positive value of α (where α and θ are in degrees) such that **1**

$$\sin(\alpha - \theta) = \sin\theta$$

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(c) Hence, show that the ratio of the area of $\triangle ABC$ to the area of $\triangle ADC$ is $\sqrt{3} : 2$. **3**

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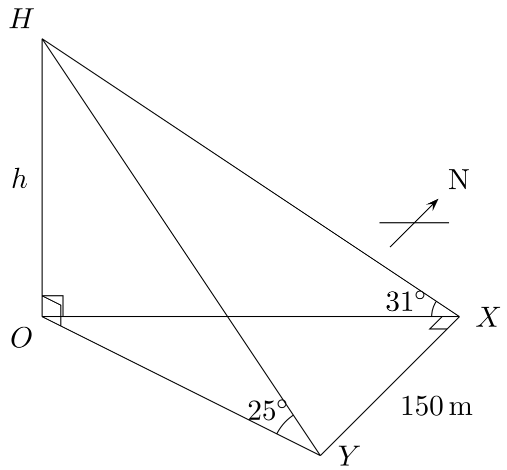
Examination continues overleaf...

Question 20 (4 marks)

From a point X due east of a tower, the angle of elevation of the top of the tower H is $31°$. From another point Y due south of X, the angle of elevation is $25°$. $XY = 150\,\text{m}$.



(a) Show that $h = \dfrac{150 \tan 25° \tan 31°}{\sqrt{\tan^2 31° - \tan^2 25°}}$. **3**

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(b) Hence, find the height of the tower. **1**

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Question 21 (4 marks)

A particle is moving such that its displacement in metres after t seconds is given by the equation

$$x = -1.5 + 3\sin 2t$$

(a) State the maximum displacement of the particle. **1**

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(b) Find the first four times when the particle is at the origin. **3**

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Examination continues overleaf...

Question 22 (9 marks)

A particle travels in a straight line. Its velocity $\dot{x}$ at time t is given by

$$\dot{x} = \left(3t^2 - 12t + 9\right) \text{ metres per second}$$

(a) Find the expression for the particle's displacement x in terms of t, if the particle was initially at $x = 2$. **2**

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(b) Find an expression for the particle's acceleration in terms of t. **1**

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(c) At what times is the particle at rest? **1**

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(d) Draw a velocity-time graph that represents the motion of this particle, showing all important features. **2**

(e) Find the total distance travelled between $t = 0$ to $t = 3$. **3**

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Examination continues overleaf...

Question 23 (3 marks)

Given $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ and using only translations, reflections and dilations, carefully state all of the transformations required to transform the function $y = \sin x$ to obtain the function $y = \sin^2 x$. **3**

Question 24 (13 marks)

Consider the function:

$$f(x) = \log_e \left(x^2 + 1\right)$$

(a) Find the stationary point(s) of $f(x)$ and determine their nature. **3**

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(b) Find the coordinates of the point(s) of inflecion of $f(x)$. **3**

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(c) Hence, sketch the curve of $f(x)$, showing all key features. **2**

(d) Use the Trapezoidal Rule with four subintervals to find an approximation to: **3**

$$\int_1^3 \log_e\left(x^2+1\right)\, dx$$

Give your answer correct to 3 decimal places.

(e) State whether the approximation found in part (d) is greater or less than the exact value of $\int_1^3 \log_e\left(x^2+1\right)\, dx$. Use your sketch in part (c) to provide a brief justification for your answer. **2**

Examination continues overleaf...

Question 25 (5 marks)

An open-topped fish tank of volume $90\,\text{m}^3$ is to be made in the shape of a rectangular prism of length $2x$ metres, width x metres, and height h metres. Materials cost \$15 per square metre for the base of the tank and \$20 per square metre for the sides of the tank.

Show that the total cost $\$C$ of making the fish tank is given by

$$C = 30x^2 + \frac{5\,400}{x}$$

and hence find the dimensions of the fish tank with the least total cost.

Question 26 (4 marks)

Mrs Bhamra and Ms Chaussivert's classes each sat twenty class tests. Ms Bhamra's class results on the tests are displayed in the box-and-whisker plot shown in part (a).

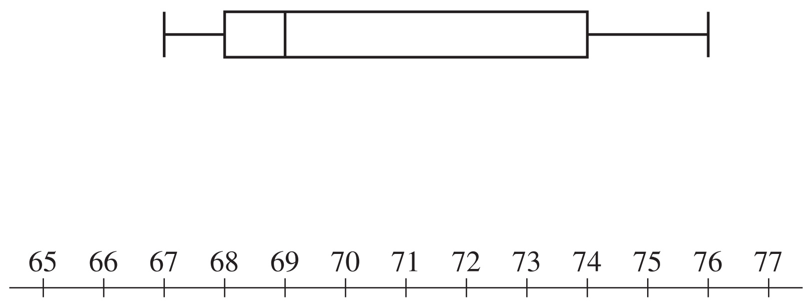
(a) Ms Chaussivert's 5-number summary for her class' test results is **1**

`67, 69, 71, 73 75`

Draw a box-and-whisker plot to display Ms Chaussivert's class results below that of Mrs Bhamra's class results.

Ms Bhamra

Ms Chaussivert

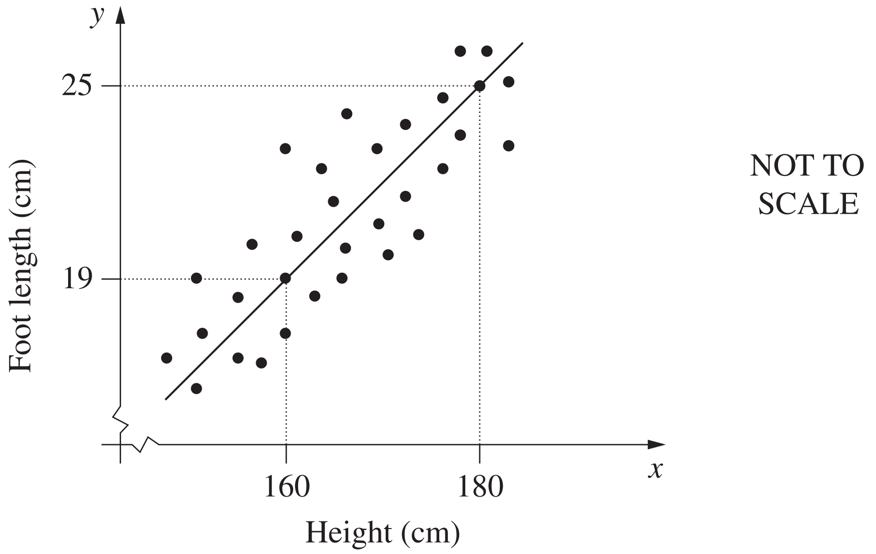


(b) Mrs Bhamra's class claims that they scored better than Ms Chaussivert's class. Are they correct? Justify your answer by referring to the summary statistics and the skewness of the distributions. **3**

Examination continues overleaf...

Question 27 (4 marks)

Each member of a group of males had his height and foot length measured and recorded. The results were graphed and a line of fit drawn.



(a) Why does the value of the y intercept have no meaning in this situation? **1**

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(b) George is 10 cm taller than his brother Harry. Use the line of fit to estimate the difference in their foot lengths. **1**

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(c) Sam calculated a correlation coefficient of -1.2 for the data. Give TWO reasons why Sam must be incorrect. **2**

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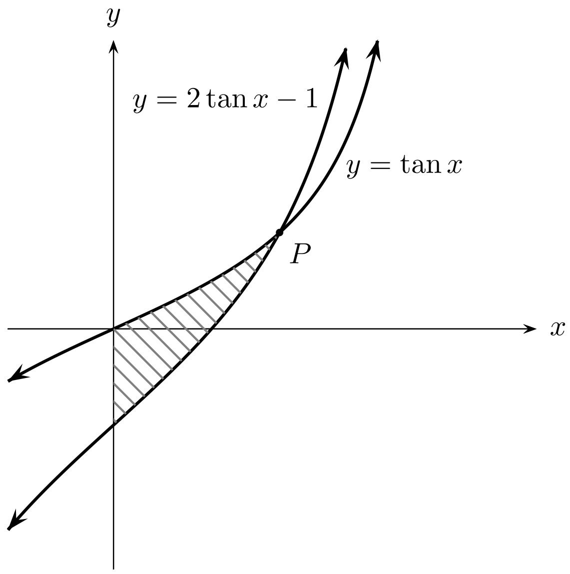
Question 28 (4 marks)

The diagram below shows a sketch of parts of the graphs of $y = \tan x$ and $y = 2\tan x - 1$. The graphs intersect at the point P. **4**



Show that the area shaded is equal to $\left(\dfrac{\pi}{4} - \dfrac{1}{2}\log_e 2\right)$ square units.

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Examination continues overleaf...

Question 29 (4 marks)

The level of a local dam is being monitored. After t days, the rate at which the volume of water in the dam is changing is given by **4**

$$R = \frac{1\,000}{1+t} - 500$$

where R is the rate of change of the volume of water in the dam, measured in megalitres per day.

The volume of water in the dam at the end of the fifth day of monitoring was half the volume at the end of the fourth day.

Find an expression for the volume of water V in the dam, t days after the monitoring began.

End of paper.

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NESA STUDENT #: Solutions

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Marker's use only.

QUESTION	1-10	11-14	15-17	18-20	21-23	24-25	26-29	**Total**
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Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

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Questions **Marks**

1. What is the solution to the equation $\log_3(2x-5) = 2$? **1**

(A) $x = 1$ (B) $x = \dfrac{13}{2}$ (C) $x = \dfrac{11}{2}$ (D) $x = 7$

2. Evaluate $\lim\limits_{x \to 7} \dfrac{x^2+5x-84}{x-7}$ **1**

(A) -5 (B) 0 (C) 12 (D) 19

3. For what values of a does the equation $ax^2+5x+a=0$ have no solution? **1**

(A) $a > 0$
(B) $a = \frac{5}{2}$
(C) $-\frac{5}{2} < a < \frac{5}{2}$
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4. A function f is given by the rule **1**

$$f(x) = \begin{cases} (x-2)^2+1, & x \le 2 \\ 4x-7, & x > 2 \end{cases}$$

What is the value of $f\Big(f(0)\Big)$?

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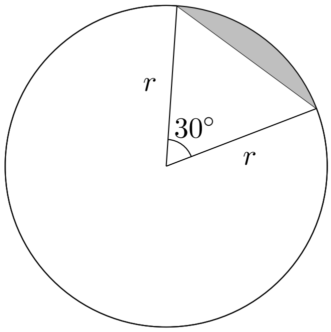
6. The diagram shows a circle with radius r centimetres and a sector which subtends an angle of $30°$ **1**



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(C) $2\tan x + c$

(B) $4\tan x \sec^2 x + c$

(D) $2x + 2\sec x + c$

8. What is the domain and range of the function $y = \frac{1}{\sqrt{x-9}}$? **1**

(A) $D = \{x : x \geq 9\}$ and $R = \{y : y > 0\}$

(B) $D = \{x : x > 9\}$ and $R = \{y : y > 0\}$

(C) $D = \{x : -\infty \leq x \leq \infty\}$ and $R = \{y : -\infty \leq y \leq \infty\}$

(D) $D = \{x : x \leq -3 \text{ or } x \geq 3\}$ and $R = \{y : y < 0\}$

9. Olivia and Sarah are participating in a fitness program. The probability that Olivia will complete the program is 0.8, and the probability that Sarah will complete the program is 0.7. **1**

What is the probability that only one of them will complete the program?

(A) 0.14 (B) 0.24 (C) 0.38 (D) 0.56

10. Which of the following is the derivative of $e^{3^x x^6}$. **1**

(A) $3^x x^6 e^{3^x x^6}$

(C) $3^x x^5 (6 + x \log_e 3) e^{3^x x^6}$

(B) $3^x x^6 e^{(3^x x^6 - 1)}$

(D) $3^x x^5 \left(3^x x^6 + 6x^5 (\log_e 3) 3^x\right) e^{3^x x^6}$

Examination continues overleaf...

Section II

90 marks
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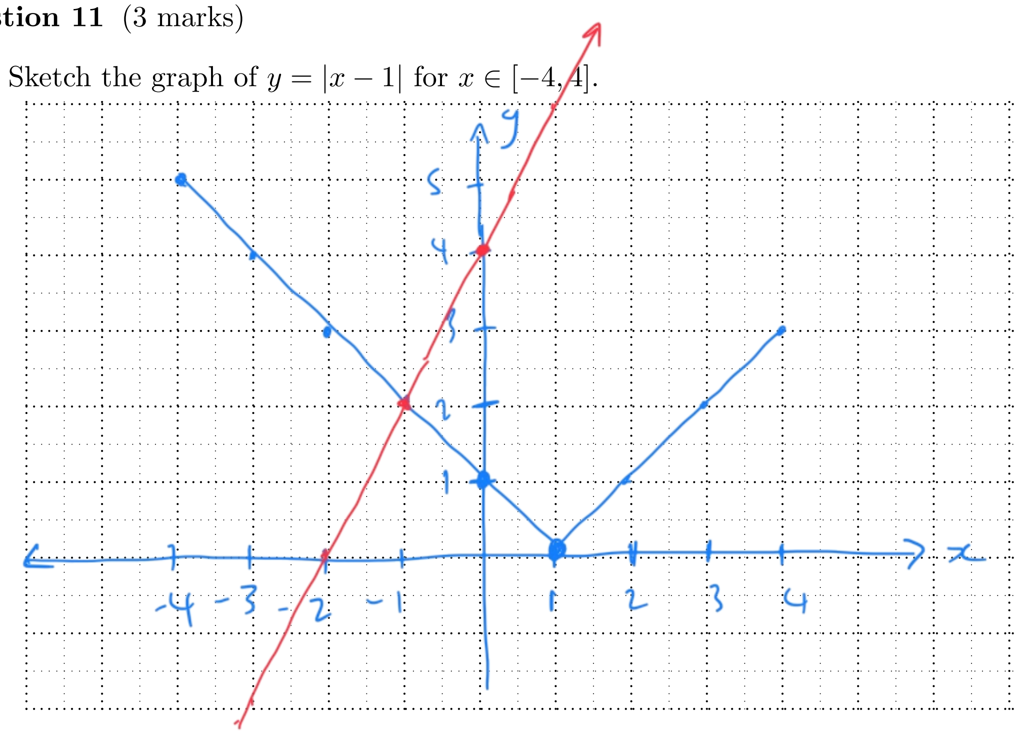
Question 11 (3 marks)

(a) Sketch the graph of $y = |x - 1|$ for $x \in [-4, 4]$. **1**



(b) Using the sketch from part (i) or otherwise, solve **2**

$$|x - 1| = 2x + 4$$

if $|x-1| \geq 0$ if $|x-1| < 0$

using the diagram OR $x - 1 = 2x + 4$ $-x - 1 = 2x + 4$

$x = -1$ ✓ $-x = 5$ $-3x = 3$

(need to have sketched $x = -5$ ✗ $x = -1$

the line $y = 2x + 4$) ✓ but $-5-1$ is not $= -1 - 1$

≥ 0 $= -2$

$-2 - 1 < 0$ ✓

$\therefore x = -1$ ✓

Question 12 (4 marks)

Given $f(x) = \sqrt{x^2 - 9}$ and $g(x) = x + 5$

(a) Find positive integers c and d such that $f\Big(g(x)\Big) = \sqrt{(x+c)(x+d)}$ **2**

$f(g(x)) = \sqrt{(x+5)^2 - 9}$ ✓

$= \sqrt{x^2 + 10x + 25 - 9}$

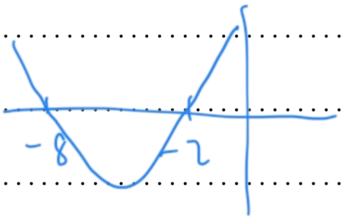
$= \sqrt{x^2 + 10x + 16}$

$= \sqrt{(x+8)(x+2)}$ ✓

$\therefore c = 8,\ d = 2$ (or vice-versa)

(b) State the domain for which $f\Big(g(x)\Big)$ is defined. **2**

$(x+8)(x+2) \geq 0$ ✓



Domain: $x \leq -8$ and $x \geq -2$ ✓

Question 13 (4 marks)

(a) Show that $\dfrac{d}{dx}(x \operatorname{cosec} x) = \operatorname{cosec} x - x \cot x \operatorname{cosec} x$ **3**

$x \operatorname{cosec} x = \dfrac{x}{\sin x}$ ✓ $u = x \quad v = \sin x$

$\dfrac{d}{dx}(x \operatorname{cosec} x) = \dfrac{\sin x - x\cos x}{\sin^2 x}$ ✓ $u' = 1 \quad v' = \cos x$

$= \dfrac{1}{\sin x} - \dfrac{x\cos x}{\sin^2 x}$ ✓

$= \operatorname{cosec} x - x\cot x \operatorname{cosec} x = \operatorname{cosec} x\,(1 - x\cot x)$

(b) Hence find: **1**

$$\int 2\Big(\operatorname{cosec} x\,(1 - x\cot x)\Big)\,dx$$

$\int 2(\operatorname{cosec} x\,(1 - x\cot x))\,dx$

$= 2\int \operatorname{cosec} x\,(1 - x\cot x)\,dx$

$= 2x\operatorname{cosec} x + C$ (from part (a)) ✓

Examination continues overleaf...

Question 14 (3 marks)

Find the equation of the normal to the curve $y = \left(x + \frac{2}{x}\right)^2$ at the point where $x = 2$. **3**
Write the equation of the normal using general form.

$y = (x + 2x^{-1})^2$ when $x = 2$, $y = \left(2 + \frac{2}{2}\right)^2 = 9$

$= x^2 + 4 + 4x^{-2}$

$\frac{dy}{dx} = 2x - 8x^{-3}$

$\therefore$ m of tangent at $x=2$ $= 2(2) - 8(2)^{-3} = 3$ ✓

$\therefore$ m of normal $= -\frac{1}{3}$ ✓

$\therefore$ equation of normal is $y - 9 = -\frac{1}{3}(x-2)$

$3y - 27 = -x + 2$

$x + 3y - 29 = 0$ ✓

Question 15 (3 marks)

A and B are two events such that $P(A) = 0.3$, $P(B) = 0.2$ and $P(A|B) = 0.5$

(a) Show that $P(A \cup B) = 0.4$. **2**

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$0.5 = \frac{P(A \cap B)}{0.2}$

$P(A \cap B) = 0.1$ ✓

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.2 - 0.1$

$= 0.4$ ✓

(b) Find the value of **1**

$$P\left(\overline{A} \cap \overline{B}\right)$$

$P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = 0.6$ ✓

Question 16 (4 marks)

The discrete random variable X has the following distribution:

x	0	1	2	3	4
$P(X = x)$	0.1	$5a^2$	0.2	0.2	$2.5 + 11a$

0.2 0.3

(a) Show that $a = -\frac{1}{5}$. **3**

$\sum P(X=x) = 1$

$0.1 + 5a^2 + 0.2 + 0.2 + 2.5 + 11a = 1$ ✓

$5a^2 + 11a + 3 = 1$

$5a^2 + 11a + 2 = 0$ $\times 10$

$5a^2 + 10a + a + 2 = 0$ $+ 11$

$5a(a+2) + (a+2) = 0$

$(5a+1)(a+2) = 0$

$a = -\frac{1}{5}, -2$ ✓

BUT $2.5 + 11 \times -2 = -19.5$ ✗ $(0 \leq P(X=x) \leq 1)$ } ✓

$\therefore a = -\frac{1}{5}$

(b) Find the mean of the distribution. **1**

$\mu = 0 \times 0.1 + 1 \times 5(-\frac{1}{5})^2 + 2 \times 0.2 + 3 \times 0.2$
$+ 4 \times (2.5 + 11 \times -\frac{1}{5})$

$= 2.4$ ✓

Examination continues overleaf...

Question 17 (6 marks)

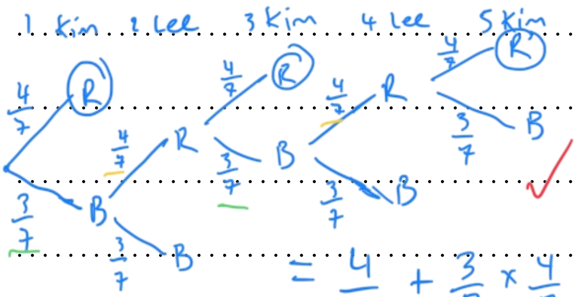
There are 4 red and 3 black discs in a bag. Ms Kim and Ms Lee are playing a game in which they take turns drawing a disc from the bag and then replacing it.

To win the game, Ms Kim must draw a red disc and for Ms Lee to win, she must draw a black disc. They continue taking turns until there is a winner. Ms Kim draws a disc first.

(a) Find the probability that Ms Kim wins on her first draw. **1**

$\frac{4}{7}$ ✓

(b) Find the probability that Ms Kim wins in three or less of her turns. **2**

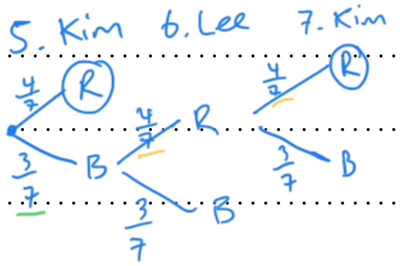


P(Ms Kim wins in 3 or less of her turns)

$= P(W) + P(LLW) + P(LLLLW)$ ✓

$= \frac{4}{7} + \frac{3}{7} \times \frac{4}{7} \times \frac{4}{7} + \frac{3}{7} \times \frac{4}{7} \times \frac{3}{7} \times \frac{4}{7} \times \frac{4}{7} = 0.75$ ✓

(c) Find the probability that Ms Kim wins the game. **3**



$P(1 \text{ turn}) = \frac{4}{7}$

$P(2 \text{ turns or less}) = \frac{4}{7} + \frac{3}{7} \times \frac{4}{7} \times \frac{4}{7} = \frac{4}{7} + \frac{3}{7}\left(\frac{4}{7}\right)^2$

$P(3 \text{ turns or less}) = \frac{4}{7} + \frac{3}{7}\left(\frac{4}{7}\right)^2 + \left(\frac{3}{7}\right)^2\left(\frac{4}{7}\right)^3$

$P(4 \text{ turns or less}) = \frac{4}{7} + \frac{3}{7}\left(\frac{4}{7}\right)^2 + \left(\frac{3}{7}\right)^2\left(\frac{4}{7}\right)^3 + \left(\frac{3}{7}\right)^3\left(\frac{4}{7}\right)^4$

$\therefore P(n \text{ turns or less})$

$= \frac{4}{7} + \frac{3}{7}\left(\frac{4}{7}\right)^2 + \ldots \left(\frac{3}{7}\right)^{n-1}\left(\frac{4}{7}\right)^n$

G.P with $r = \frac{3}{7} \times \frac{4}{7}$ ✓

P(Ms Kim wins the game) = limiting sum of the GP.

$S_\infty = \dfrac{\frac{4}{7}}{1 - \frac{3}{7} \times \frac{4}{7}}$ ✓ $= \dfrac{28}{37}$ ✓

Question 18 (2 marks)

(a) Prove the following identity: **2**

$$\frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{\cos\theta} = 2\sec\theta.$$

$$LHS = \frac{\cos^2\theta + (1+\sin\theta)^2}{\cos\theta(1+\sin\theta)}$$

$$= \frac{1-\sin^2\theta + 1 + 2\sin\theta + \sin^2\theta}{\cos\theta(1+\sin\theta)}$$ ✓

$$= \frac{2+2\sin\theta}{\cos\theta(1+\sin\theta)}$$

$$= \frac{2(1+\sin\theta)}{\cos\theta(1+\sin\theta)}$$

$$= \frac{2}{\cos\theta}$$

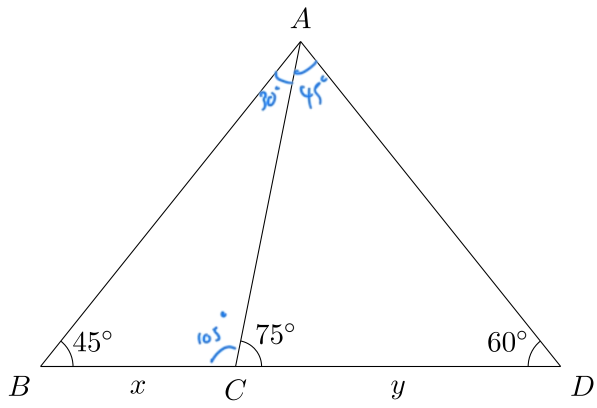
$$= 2\sec\theta$$ ✓

$$= RHS$$

Examination continues overleaf...

Question 19 (7 marks)

Refer to the following diagram:



(a) Show that $\frac{x}{y} = \frac{\sqrt{3}}{2}$. **3**

$\frac{AC}{\sin 45} = \frac{x}{\sin 30}$ $\frac{AC}{\sin 60} = \frac{y}{\sin 45}$

$\frac{AC}{\frac{1}{\sqrt{2}}} = \frac{x}{\frac{1}{2}}$ $\frac{AC}{\frac{\sqrt{3}}{2}} = \frac{y}{\frac{1}{\sqrt{2}}}$

$\sqrt{2}AC = 2x$ $\frac{2AC}{\sqrt{3}} = \sqrt{2}y$

$AC = \frac{2x}{\sqrt{2}}$ ① ✓

$AC = \frac{\sqrt{6}y}{2}$ ② ✓

sub ① into ②: $\frac{2x}{\sqrt{2}} = \frac{\sqrt{6}y}{2}$

$\frac{x}{y} = \frac{\sqrt{12}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$ ✓

(b) Find the smallest positive value of α (where α and θ are in degrees) such that **1**

$$\sin(\alpha - \theta) = \sin\theta$$

180° ✓

(c) Hence, show that the ratio of the area of $\triangle ABC$ to the area of $\triangle ADC$ is $\sqrt{3} : 2$. **3**

$\text{Area}_{\triangle ABC} = \frac{1}{2} \times x \times AC \times \sin(105)$

$\text{Area}_{\triangle ADC} = \frac{1}{2} \times AC \times y \times \sin(75)$

$\therefore \frac{A_{\triangle ABC}}{A_{\triangle ADC}} = \frac{\frac{1}{2} x AC \sin(105)}{\frac{1}{2} y AC \sin(75)}$ ✓

$= \frac{x \sin(105)}{y \sin(75)}$

$= \frac{x \sin(180-75)}{y \sin(75)}$ ✓

$= \frac{x \sin 75}{y \sin 75}$

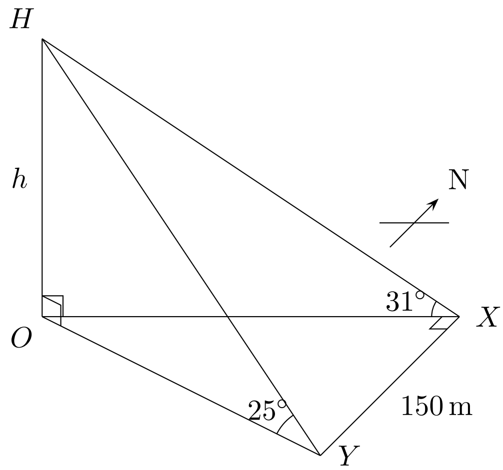
$= \frac{x}{y} = \frac{\sqrt{3}}{2}$

$\therefore A_{\triangle ABC} : \triangle_{\triangle ADC} = \sqrt{3} : 2$ ✓

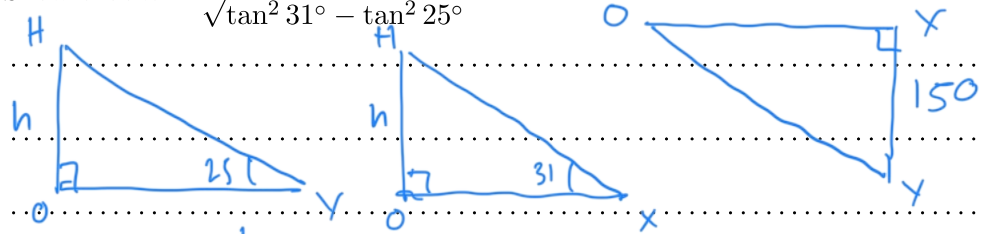
Examination continues overleaf...

Question 20 (4 marks)

From a point X due east of a tower, the angle of elevation of the top of the tower H is $31°$. From another point Y due south of X, the angle of elevation is $25°$. $XY = 150\,\text{m}$.



(a) Show that $h = \dfrac{150 \tan 25° \tan 31°}{\sqrt{\tan^2 31° - \tan^2 25°}}$. **3**



$\tan 25 = \dfrac{h}{OY}$ $\tan 31 = \dfrac{h}{OX}$

$OY = \dfrac{h}{\tan 25}$ $OX = \dfrac{h}{\tan 31}$

$150^2 + OX^2 = OY^2$

$150^2 + \left(\dfrac{h}{\tan 31}\right)^2 = \left(\dfrac{h}{\tan 25}\right)^2$

$150^2 + \dfrac{h^2}{\tan^2 31} = \dfrac{h^2}{\tan^2 25}$

$h^2\left(\dfrac{1}{\tan^2 25} - \dfrac{1}{\tan^2 31}\right) = 150^2$

$h^2\left(\dfrac{\tan^2 31 - \tan^2 25}{\tan^2 25 \tan^2 31}\right) = 150^2$

$h^2 = \dfrac{150^2 \tan^2 25 \tan^2 31}{\tan^2 31 - \tan^2 25}$

$h = \dfrac{150 \tan 25 \tan 31}{\sqrt{\tan^2 31 - \tan^2 25}}$

(b) Hence, find the height of the tower. **1**

$h = 110.91\,\text{m}$

Question 21 (4 marks)

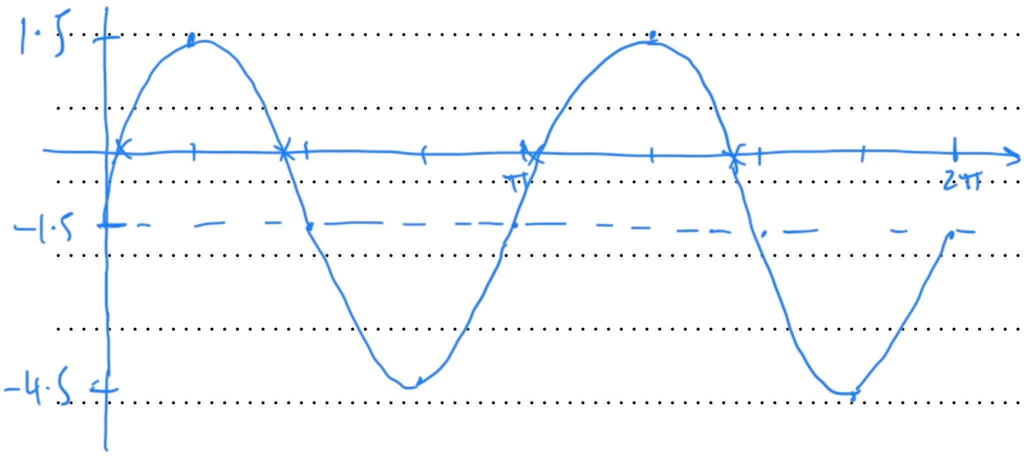
A particle is moving such that its displacement in metres after t seconds is given by the equation

$$x = -1.5 + 3\sin 2t$$

(a) State the maximum displacement of the particle. **1**

$-1.5 + 3 = +1.5$ m ✓

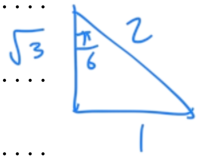
(b) Find the first four times when the particle is at the origin. **3**



let $x = 0$: $-1.5 + 3\sin(2t) = 0$ ✓

$3\sin(2t) = 1.5$

$\sin(2t) = \frac{1}{2}$



$2t = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$ ✓

$2t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

∴ the first four times the particle is at the origin are:

$t = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$ seconds. ✓

Examination continues overleaf...

Question 22 (9 marks)

A particle travels in a straight line. Its velocity $\dot{x}$ at time t is given by

$$\dot{x} = \left(3t^2 - 12t + 9\right) \text{ metres per second}$$

(a) Find the expression for the particle's displacement x in terms of t, if the particle was initially at $x = 2$. **2**

$x = \int \dot{x}\, dt = \int 3t^2 - 12t + 9\, dt$

$= t^3 - 6t^2 + 9t + c$ ✓

sub in $t = 0$, $x = 2$

$2 = 0^3 - 6(0)^2 + 9(0) + c \Rightarrow c = 2$

$\therefore$ displacement is given by $x = t^3 - 6t^2 + 9t + 2$ metres ✓

(b) Find an expression for the particle's acceleration in terms of t. **1**

$\ddot{x} = \frac{d}{dt}(\dot{x}) = 6t - 12\ m/s^2$ ✓

(c) At what times is the particle at rest? **1**

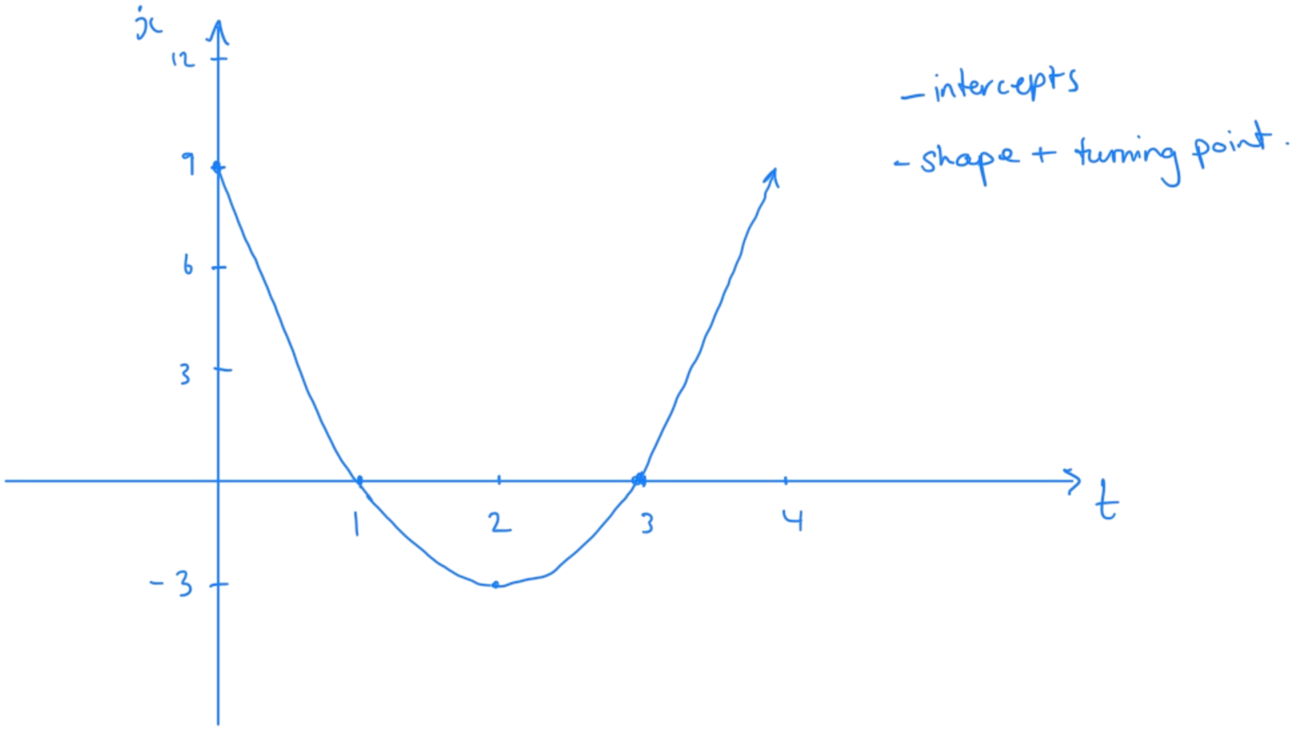
$3t^2 - 12t + 9 = 0$

$t^2 - 4t + 3 = 0$

$(x-3)(x-1) = 0$

$x = 1, 3$ seconds ✓

(d) Draw a velocity-time graph that represents the motion of this particle, showing all important features. **2**



(e) Find the total distance travelled between $t = 0$ to $t = 3$. **3**

total distance between 0 and 3 = area under the velocity curve.

$$\text{distance} = \int_0^1 3t^2 - 12t + 9\,dt + \left| \int_1^3 3t^2 - 12t + 9\,dt \right| \text{ metres}$$

$$= \left[t^3 - 6t^2 + 9t \right]_0^1 + \left| \left[t^3 - 6t^2 + 9t \right]_1^3 \right|$$

$$= (1 - 6 + 9) - (0) + \left| (27 - 54 + 27) - (1 - 6 + 9) \right|$$

$$= 4 + |-4| = 8 \text{ metres}$$

Examination continues overleaf...

Question 23 (3 marks)

Given $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ and using only translations, reflections and dilations, carefully state all of the transformations required to transform the function $y = \sin x$ to obtain the function $y = \sin^2 x$. **3**

$y = \sin x \rightarrow y = \frac{1}{2} - \frac{1}{2}\cos(2x)$

- $\sin(x) \rightarrow \cos(x)$ since $\cos(x) = \sin(\frac{\pi}{2} - x)$ this step requires a horizontal translation to the right by $\frac{\pi}{2}$ ✓

- $\cos(x) \rightarrow \cos(2x)$ this step requires a horizontal compression by a factor of 2.

- $\cos(2x) \rightarrow -\frac{1}{2}\cos(2x)$ this step requires a vertical reflection and a vertical compression by a factor of 2. ✓

- $-\frac{1}{2}\cos(2x) \rightarrow \frac{1}{2} - \frac{1}{2}\cos(2x)$ this step requires a vertical translation of $\frac{1}{2}$ in the upwards direction. ✓

Question 24 (13 marks)

Consider the function:

$$f(x) = \log_e\left(x^2 + 1\right)$$

(a) Find the stationary point(s) of $f(x)$ and determine their nature. **3**

Stat. points: let $f'(x) = 0$

$f'(x) = \frac{2x}{x^2+1} = 0 \rightarrow 2x = 0$

$x = 0$ is a stationary point

$f(0) = \ln(1) = 0$ ✓

$f''(x) = \frac{2(x^2+1) - 4x^2}{(x^2+1)^2}$ $u = 2x$ $v = x^2+1$, $u' = 2$ $v' = 2x$

$f''(0) = \frac{2(0+1) - 0}{(0+1)^2} = 2$ ∪

$\therefore (0,0)$ is a local minimum stationary point. ✓

(b) Find the coordinates of the point(s) of inflecion of $f(x)$. **3**

let $f''(x) = 0$

part (a): $f''(x) = \frac{2(x^2+1) - 4x^2}{(x^2+1)^2} = 0$ ✓

$-2x^2 + 2 = 0$

$2x^2 = 2$

$x^2 = 1$

$x = \pm 1$ are possible POIs ✓

$f(1) = \ln(2)$, $f(-1) = \ln 2$

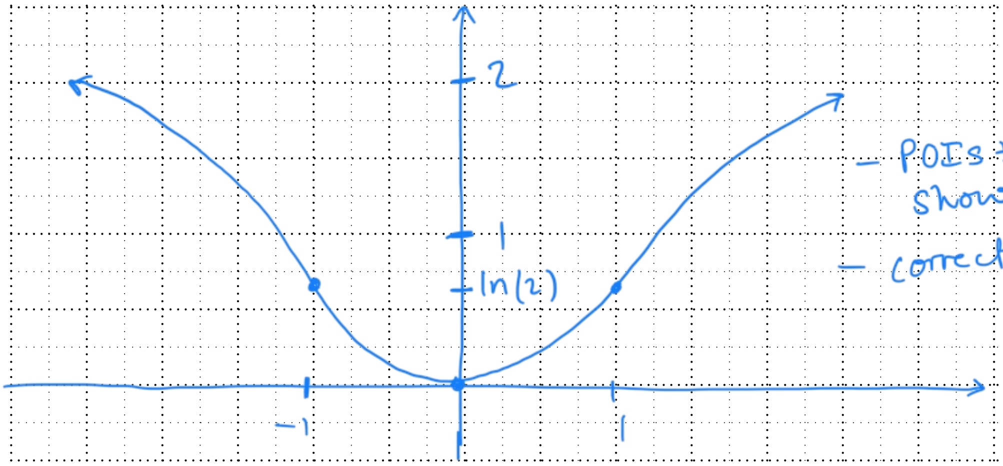
test

x	-2	-1	0	1	2
$f''(x)$	$-\frac{6}{25}$	0	2	0	$-\frac{6}{25}$
	∩	·	∪	·	∩

✓

$\therefore (-1, \ln 2)$ and $(1, \ln 2)$ are points of inflexion

(c) Hence, sketch the curve of $f(x)$, showing all key features. **2**



– POIs + Stat points shown. (1)
– correct shape. (1)

(d) Use the Trapezoidal Rule with four subintervals to find an approximation to: **3**

$$\int_1^3 \log_e\left(x^2+1\right)\,dx$$

x	1	1.5	2	2.5	3
$f(x)$	$\ln(2)$	$\ln\left(\frac{13}{4}\right)$	$\ln(5)$	$\ln\left(\frac{29}{4}\right)$	$\ln(10)$

✓

$\int_1^3 \log_e(x^2+1)\,dx$

$\simeq \frac{0.5}{2}\left(\ln(2)+\ln(10)+2\left(\ln\left(\frac{13}{4}\right)+\ln(5)+\ln\left(\frac{29}{4}\right)\right)\right)$ ✓

$\simeq 3.13$ ✓

(e) State whether the approximation found in part (b) is greater or less than the exact value of $\int_1^3 \log_e\left(x^2+1\right)\,dx$. Use your sketch in part (c) to provide a brief justification for your answer. **2**

the approximation will be less than the exact value ✓
since the curve is concave down after $x=1$ ✓,
the trapeziums will have an area slightly less
than the area under the curve for each
sub-interval.

Examination continues overleaf...

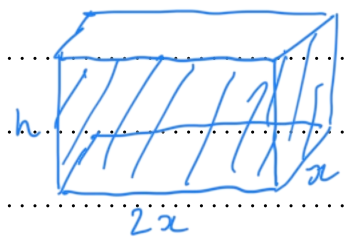
Question 25 (5 marks)

An open-topped fish tank of volume $90\,\text{m}^3$ is to be made in the shape of a rectangular prism of length $2x$ metres, width x metres, and height h metres. Materials cost \$15 per square metre for the base of the tank and \$20 per square metre for the sides of the tank.

Show that the total cost $\$C$ of making the fish tank is given by

$$C = 30x^2 + \frac{5\,400}{x}$$

and hence find the dimensions of the fish tank with the least total cost.



area of base : $2x \times x = 2x^2$

$\therefore$ cost of base : $15 \times 2x^2$

$= \$30x^2$ ✓

$V = h \times 2x \times x$

$90 = 2x^2 h$

$h = \frac{45}{x^2}$

area of sides: $2 \times h \times 2x + 2 \times x \times h$

$= 4xh + 2xh = 6xh$

$= \frac{270}{x}$

cost of sides $= 20 \times \frac{270}{x} = \$\frac{5400}{x}$ ✓

$\therefore$ total cost of tank, $C = 30x^2 + \frac{5400}{x}$ ✓

trying to minimise cost: let $\frac{dC}{dx} = 0$

$\frac{dC}{dx} = 60x - \frac{5400}{x^2} = 0$

$60x = \frac{5400}{x^2}$

$x^3 = 90$

$x = \sqrt[3]{90}$ ✓

check that $\sqrt[3]{90}$ is a minimum point.

$\frac{d^2C}{dx^2} = 60 + \frac{10800}{x^3} \Big|_{x=\sqrt[3]{90}}$

$= 2469\ldots$ ∪

$\therefore$ the minimum cost of the tank is when $x = \sqrt[3]{90}$

$\therefore$ dimensions of the tank are length $= 2\sqrt[3]{90}$ m

width $= \sqrt[3]{90}$ m

height $= \frac{45}{(\sqrt[3]{90})^2} = \frac{45}{90^{\frac{2}{3}}}$ m. ✓

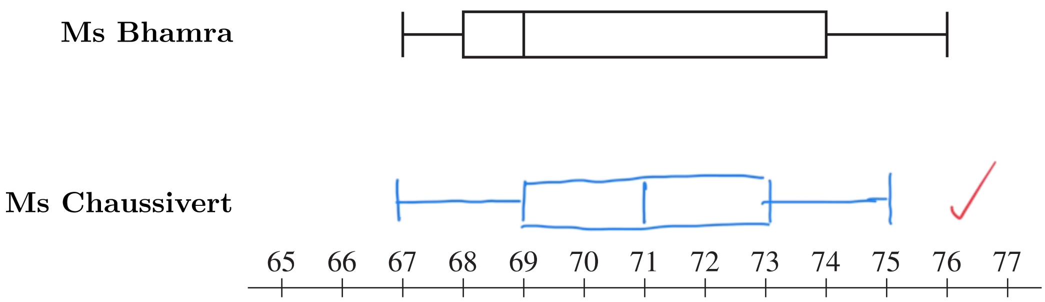
Question 26 (4 marks)

Mrs Bhamra and Ms Chaussivert's classes each sat twenty class tests. Ms Bhamra's class results on the tests are displayed in the box-and-whisker plot shown in part (a).

(a) Ms Chaussivert's 5-number summary for her class' test results is **1**

67, 69, 71, 73 75

Draw a box-and-whisker plot to display Ms Chaussivert's class results below that of Mrs Bhamra's class results.



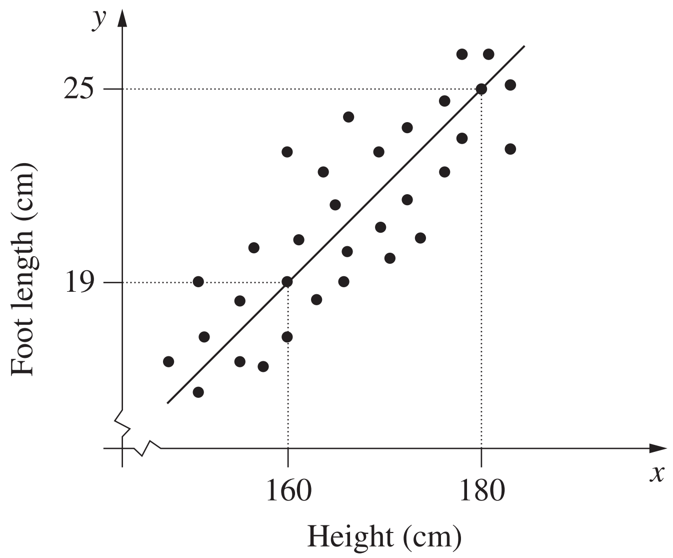
(b) Mrs Bhamra's class claims that they scored better than Ms Chaussivert's class. Are they correct? Justify your answer by referring to the summary statistics and the skewness of the distributions. **3**

Mrs Bhambra's class is wrong ✓ Miss Chaussirents class had a much higher median score and were not skewed to the right, unlike Mrs Bhamra's class, meaning they performed better on average ✓. Ms Chaussivents Class also had a smaller range and IQR, meaning they were more consistent. ✓

Examination continues overleaf...

Question 27 (4 marks)

Each member of a group of males had his height and foot length measured and recorded. The results were graphed and a line of fit drawn.



NOT TO SCALE

(a) Why does the value of the y intercept have no meaning in this situation? **1**

the y-intercept is not within the range of the data collected, and hence it cannot be used to draw any conclusions (or similar) ✓

(b) George is 10 cm taller than his brother Harry. Use the line of fit to estimate the difference in their foot lengths. **1**

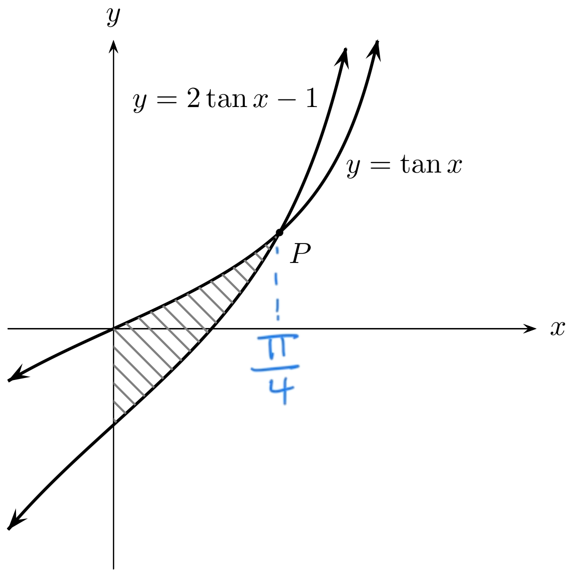
20cm difference in height predicts a 6cm difference in foot length
∴ George will have a foot length approximately 3cm longer than Harry ✓

(c) Sam calculated a correlation coefficient of -1.2 for the data. Give TWO reasons why Sam must be incorrect. **2**

1. Person's correlation coefficient must be a number between −1 and 1 so Sam must have calculated it incorrectly. ✓
2. the graph clearly shows a posstive relationship between height and foot length, so −1.2 must be wrong. ✓

Question 28 (4 marks)

The diagram below shows a sketch of parts of the graphs of $y = \tan x$ and $y = 2\tan x - 1$. The graphs intersect at the point P. **4**



Show that the area shaded is equal to $\left(\frac{\pi^2}{32} - \frac{1}{2}\log_e 2\right)$ square units.

find P : $2\tan x - 1 = \tan x$

$\tan x = 1$

$x = \frac{\pi}{4}$ (1st solution) ✓

Area $= \int_0^{\frac{\pi}{4}} \tan x - (2\tan x - 1)\,dx$ ✓

$= \int_0^{\frac{\pi}{4}} -\tan x + 1\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x} + 1\,dx = \left[\ln(\cos x) + x\right]_0^{\frac{\pi}{4}}$ ✓

$= \left(\ln\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4}\right) - \left(\ln(1) - 0\right)$

$= \ln\left(2^{-\frac{1}{2}}\right) + \frac{\pi}{4}$

$= -\frac{1}{2}\ln 2 + \frac{\pi}{4}$ ✓

Examination continues overleaf...

Question 29 (4 marks)

The level of a local dam is being monitored. After t days, the rate at which the volume of water in the dam is changing is given by **4**

$$R = \frac{1\,000}{1+t} - 500$$

where R is the rate of change of the volume of water in the dam, measured in megalitres per day.

The volume of water in the dam at the end of the fifth day of monitoring was half the volume at the end of the fourth day.

Find an expression for the volume of water V in the dam, t days after the monitoring began.

$V = \int R\,dt = \int \frac{1000}{1+t}\,dt - \int 500\,dt$

$= 1000\int \frac{1}{1+t}\,dt - 500t + c$

$V = 1000\ln|1+t| - 500t + c$ ✓

$V(4) = 2V(5)$

$1000\ln 5 - 2000 + c = 2(1000\ln 6 - 2500 + c)$ ✓

$1000\ln 5 - 2000 + c = 2000\ln 6 - 5000 + 2c$

$1000\ln 5 - 2000 - 2000\ln 6 + 5000 = c$

$\therefore c = 1000(\ln 5 - 2\ln 6) + 3000$

$= 1000\left(\ln \frac{5}{36}\right) + 3000$

$= 1000\left(\ln \frac{5}{36} + 3\right)$ ✓

$\therefore V = 1000\ln|1+t| - 500t + 1000\left(\ln\frac{5}{36} + 3\right)$ ✓

End of paper.

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

NESA STUDENT #: ....

Class (please ✔)

◯ 12MAA.1 – Miss Chaussivert

◯ 12MAX.1 – Miss J. Kim

◯ 12MAX.2 – Mrs Bhamra

◯ 12MAX.3 – Miss Lee

Directions for multiple choice answers

- Read each question and its suggested answers.
- Select the alternative Ⓐ, Ⓑ, Ⓒ, or Ⓓ that best answers the question.
- Mark only one circle per question. There is only *one* correct choice per question.
- Fill in the response circle completely, using blue or black pen, e.g.

  Ⓐ Ⓑ ● Ⓓ

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

  Ⓐ Ⓑ ⊗ ●

- If you continue to change your mind, write the word `correct` and clearly indicate your final choice with an arrow as shown below:

  Ⓐ Ⓑ ⊗ ⊗ ← correct

1 –	Ⓐ	Ⓑ	Ⓒ	●
2 –	Ⓐ	Ⓑ	Ⓒ	●
3 –	Ⓐ	Ⓑ	●	Ⓓ
4 –	Ⓐ	Ⓑ	Ⓒ	●
5 –	●	Ⓑ	Ⓒ	Ⓓ
6 –	Ⓐ	Ⓑ	●	Ⓓ
7 –	Ⓐ	Ⓑ	●	Ⓓ
8 –	Ⓐ	●	Ⓒ	Ⓓ
9 –	Ⓐ	Ⓑ	●	Ⓓ
10 –	Ⓐ	Ⓑ	●	Ⓓ